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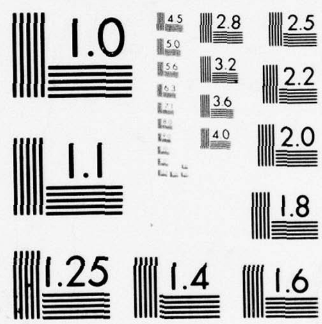

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USE OF FLOYD'S ALGORITHM TO FIND SHORTEST RESTRICTED PATHS,

by

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Victor Klee ~~and~~ David Larman

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USE OF FLOYD'S ALGORITHM TO FIND SHORTEST RESTRICTED PATHS

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Abstract. In a directed network with no negative circuit, Floyd's algorithm finds, for each pair of nodes x and y, a shortest path from x to y. Here the procedure is extended to minimize more general length-functions over sets of paths that are restricted in various ways.

Introduction.

Throughout this paper, G denotes a complete directed graph with n nodes and $n^2 - n$ edges. Each edge is an ordered pair (i, j) of nodes and has as its length a number $\lambda(i, j) \in R^* =]-\infty, \infty]$. For notational convenience G's node-set is assumed to be the set $N = \{1, \cdots, n\}$.

A walk from x to y is a node-sequence $(x_0, \cdots, x_t)$ such that $x_0 = x$, $x_t = y$, and $t > 0$. It is a chain if no node is repeated and a circuit if $x_t = x_0$ but there is otherwise no repetition. Both chains and circuits are called paths, a practice that is unusual but is convenient for our purposes.

Floyd's algorithm [R][W][F][H2][L] initializes $S[i, j] \leftarrow \lambda(i, j)$ for all $i, j \in N$ and then proceeds as follows:

```
for k ← 1 until n do
  for i ← 1 until n do
    for j ← 1 until n do
      S[i, j] ← min{S[i, j], S[i, k] + S[k, j]}.
```

If there are no circuits of negative length then $S[x, y]$ emerges as the length of a shortest path from x to y. The computation is easily modified to find shortest paths in addition to their lengths.

In the present paper the procedure is extended to deal with a family $\underline{\underline{F}}$ of sets of walks and with a walk-length function L more general than the usual one. Under suitable assumptions the extended procedure finds, for each choice of $x, y \in N$ and $\underset{\sim}{Z} \in \underline{\underline{F}}$, a shortest $\underset{\sim}{Z}$-path from x to y. That is, L is minimized over the set of all paths from x to y that belong to $\underset{\sim}{Z}$. In the "classical" case, $\underset{\sim}{Z}$ is the set of all paths (or walks) in G, $\underline{\underline{F}} = \{\underset{\sim}{Z}\}$, and the length of a walk is the sum of the length of its edges.

The Assumptions.

The function L is used to measure the length of a walk in terms of the lengths of its edges. It is assumed the range of L is contained in R^*, the domain of L is the set of all finite sequences in R^*, and the following two conditions are satisfied:

(1) if $\alpha_1, \cdots, \alpha_t \in R^*$ and $0<s<t$ then

$$L(\alpha_1, \cdots, \alpha_t) = L(L(\alpha_1, \cdots, \alpha_s), L(\alpha_{s+1}, \cdots, \alpha_t))$$

(2) if $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \text{rng } L$ with $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$ then $L(\alpha_1, \alpha_2) \leq L(\beta_1, \beta_2)$.

The length of a walk $W = (x_0, \cdots, x_t)$ is defined as

$$L_\lambda(W) = L(\lambda(x_0, x_1), \cdots, \lambda(x_{t-1}, x_t)).$$

By (1), $L_\lambda(UV) = L_\lambda(L_\lambda(U), L_\lambda(V))$ when U is a walk from x to y and V is a walk from y to z. Here UV denotes the walk that follows U from x to y and then follows V from y to z.

Among the admissible functions L are

$$L_p(\alpha_1, \cdots, \alpha_t) = (\alpha_1^p + \cdots + \alpha_t^p)^{1/p} \text{ for an integer } p > 0,$$

$$L_\infty(\alpha_1, \cdots, \alpha_t) = \max(\alpha_1, \cdots, \alpha_t),$$

$$L^p(\alpha_1, \cdots, \alpha_t) = (|\alpha_1|^p + \cdots + |\alpha_t|^p)^{1/p} \text{ for a real } p > 0.$$

The usual L is L_1. The function L_∞ is also of practical interest, for if G is initially equipped with nonnegative real edge-weights $\gamma(i, j)$ representing flow capacities and if $\lambda(i, j) = -\gamma(i, j)$ for all i and j, then the shortest paths with respect to L_∞ are those of maximum flow capacity for specified initial and terminal nodes [H1][H2][L].

When $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are sets of walks let

$$\underset{\sim}{U}\underset{\sim}{V} = \{UV\colon U\in\underset{\sim}{U},\ V\in\underset{\sim}{V},\ U \text{ ends where } V \text{ starts}\}.$$

Thus a walk $(x_0,\cdots, x_t)$ belongs to $\underset{\sim}{U}\underset{\sim}{V}$ if and only if there exists s such that $0<s<t$, $(x_0,\cdots, x_s)\in\underset{\sim}{U}$ and $(x_s,\cdots, x_t)\in\underset{\sim}{V}$. The first assumption about $\underline{\underline{F}}$ is:

(3) *if* $0<s<t$ *and* $(x_0,\cdots, x_t)\in\underset{\sim}{W}\in\underline{\underline{F}}$ *then there exist* $\underset{\sim}{U}$, $\underset{\sim}{V}$ *such that* $(x_0,\cdots, x_s)\in\underset{\sim}{U}\in\underline{\underline{F}}$, $(x_s,\cdots, x_t)\in\underset{\sim}{V}\in\underline{\underline{F}}$, and $\underset{\sim}{U}\underset{\sim}{V} \subset \underset{\sim}{W}$.

For a simple but interesting example, suppose that a set of special edges of G is given, and for $0\le k\le \ell$ let $\underset{\sim}{W}(k)$ denote the set of all walks $(x_0,\cdots,x_t)$ such that (x_{i-1}, x_i) is special for at most k values of i. Let $\underline{\underline{F}} = \{\underset{\sim}{W}(k) = 0\le k\le \ell\}$. Note that $\underset{\sim}{W}(i)\underset{\sim}{W}(j) \subset \underset{\sim}{W}(k)$ when $i+j\le k$.

The *edges* of a walk $W = (x_0,\cdots, x_t)$ are $(x_0, x_1),\cdots, (x_{t-1}, x_t)$. A path P is *associated* with W if P also starts at x_0 and ends at x_t, P is a subsequence of W, and each edge of P is an edge of W. It is assumed $\underline{\underline{F}}$, L and λ are interrelated as follows:

(4) *if* $W\in\underset{\sim}{W}\in\underline{\underline{F}}$ *and* P *is a path associated with* W *then* $P\in\underset{\sim}{W}$ *and* $L_\lambda(P)\le L_\lambda(W)$.

Since there are only finitely many paths in G, a consequence of (4) is:

(5) *For each* $x, y\in N$ *and* $\underset{\sim}{Z}\in\underline{\underline{F}}$, *either there is no* $\underset{\sim}{Z}$*-walk from* x *to* y *or there is a* $\underset{\sim}{Z}$*-path which is a shortest* $\underset{\sim}{Z}$*-walk from* x *to* y.

As is shown in the last section of the paper, a number of problems on shortest restricted paths can be formulated and efficiently solved in terms of a function L and a family $\underline{\underline{F}}$ of sets of walks satisfying conditions (1) - (4).

The Algorithms.

The extended version of Floyd's algorithm (EVFA) starts with the $n \times n$ matrix λ of edge-lengths of G, procedures for computing $L(\alpha)$ and $L(\alpha, \beta)$ for all $\alpha, \beta \in R^*$, and a suitable representation of the family F of sets of walks. Also required is a set T of triples (U, V, W) of members of F such that:

(6) $UV \subset W$ <u>for each</u> $(U, V, W) \in T$;

(7) <u>if</u> $0<s<t$ <u>and</u> $(x_0, \cdots, x_t) \in W \in F$ <u>then there exist</u> U <u>and</u> V <u>such that</u> $(x_0, \cdots, x_s) \in U$, $(x_s, \cdots, x_t) \in V$, <u>and</u> $(U, V, W) \in T$.

Of course T may be taken as the set of all triples (U, V, W) of members of F such that $UV \subset W$, but it is most efficient to have T as small as possible subject to (6) and (7). (In the example following (3) in the preceding section, it would be best to let $T = \{(W(i), W(j), W(k)): i+j = k \leq \ell\}$ rather than using $i+j \leq k$.)

For each $Z \in F$ the algorithm outputs four $n \times n$ matrices: an R^*-valued S_Z, an integer-valued M_Z, an F-valued U_Z and an F-valued V_Z. At the time of output these satisfy the following conditions for all $x, y \in N$:

(8) <u>if there is no</u> Z<u>-path from</u> x <u>to</u> y <u>then</u> $S_Z[x, y] = \infty$ <u>and</u> $M_Z[x, y] = -1$;

(9) <u>if there is a</u> Z<u>-path from</u> x <u>to</u> y <u>then</u>

(a) $S_Z[x, y]$ <u>is the (possibly</u> ∞<u>) length of a shortest</u> Z<u>-path from</u> x <u>to</u> y;

(b) $M_Z[x, y] = 0$ <u>if</u> (x, y) <u>is such a shortest path</u>; <u>otherwise</u>, $M_Z[x, y]$ <u>is the index</u> m <u>of an intermediate node on such a shortest path, and the path itself is formed from a shortest</u> $U_Z[x, y]$<u>-path from</u> x <u>to</u> m <u>followed by a shortest</u> $V_Z[x, y]$<u>-path from</u> m <u>to</u> y.

Using the output M, U and V of EVFA the path-tracing algorithm (PTA) actually finds the shortest paths.

EVFA: EXTENDED VERSION OF FLOYD'S ALGORITHM

```
begin
  for i ← 1 until n do
    for j ← 1 until n do
      for each W∈F do
        if (i, j)∈W then begin
                           S_W[i, j] ← L(λ(i, j));
                           M_W[i, j] ← 0
                         end
                    else begin
                           S_W[i, j] ← ∞;
                           M_W[i, j] ← -1
                         end of initialization;
  for k ← 1 until n do
    for i ← 1 until n do
      for j ← 1 until n do
        for each (U, V, W)∈T do
          if (L(S_U[i, k], S_V[k, j]) < S_W[i, j])
            or (M_W[i, j] = -1 and M_U[i, k] ≠ -1 and M_V[k, j] ≠ -1)
              then begin
                     S_W[i, j] ← L(S_U[i, k], S_V[k, j]);
                     M_W[i, j] ← k;
                     U_W[i, j] ← U;
                     V_W[i, j] ← V
                   end of main loop;
```

```
for each W∈F do begin
                    print S_W, M_W, U_W and V_W
                end
end
```

In the path-tracing algorithm, STACK's members are alternately node-indices and members of $\underline{\underline{F}}$. (It is often convenient in practice to represent the members of $\underline{\underline{F}}$ by negative integers.) When a shortest $\underset{\sim}{Z}$-path from x to y is desired, STACK is initialized as $(y, \underset{\sim}{Z}, x)$. As STACK is processed, node-indices are added to PATH, which emerges as a shortest $\underset{\sim}{Z}$-path from x to y. (PTA has no output when there is no $\underset{\sim}{Z}$-path from x to y.)

PTA: PATH-TRACING ALGORITHM

```
begin
  if M_Z[x, y] = -1 then goto NONE;
  STACK[1] ← y; STACK[2] ← Z; STACK[3] ← x;
  s ← 3; p ← 0;
  while s ≥ 3 do
    begin
      i ← STACK[s];
      W ← STACK[s-1];
      j ← STACK[s-2];
      m ← M_W[i, j];
      if m = 0 then begin
                      p ← p+1;
                      PATH[p] ← i;
                      s ← s-2
```

```
                end
            else begin
                    STACK[s-1] ← V_W[i, j];
                    STACK[s] ← m;
                    STACK[s+1] ← U_W[i, j];
                    s ← s+2;
                    STACK[s] ← i
                end
    end of loop;
  PATH[p+1] ← STACK[1];
  print PATH;
  NONE:
end
```

In describing the efficiency of EVFA and PTA we use the uniform cost criterion in the RAM model of random access computation [AHU]. Initialization of EVFA requires time $O(|\underline{\underline{F}}|n^2)$ and each passage through the main loop requires time $O(|\underline{\underline{T}}|n^2)$, so the overall time-complexity of EVFA is

$$O(|\underline{\underline{F}}|n^2 + |\underline{\underline{T}}|n^3).$$

This counts each evaluation of $L(\alpha)$ or $L(\alpha, \beta)$ as a single step.

For each choice of $x, y, \underset{\sim}{Z}$, PTA requires time $O(n)$ to find a shortest $\underset{\sim}{Z}$-path from x to y. Thus, starting from the output of EVFA, time $O(|\underline{\underline{F}}|n^2)$ is required to find, for all $x, y \in N$ and $\underset{\sim}{Z} \in \underline{\underline{F}}$, a shortest $\underset{\sim}{Z}$-path from x to y.

It remains to show that EVFA and PTA compute what is claimed for them. In the case of PTA, this follows from a routine inductive argument showing that at

the end of the initialization and also at the end of each passage through the loop there exists a shortest $\underset{\sim}{Z}$-path P from x to y such that the following three conditions are satisfied:

(a) s is odd; STACK[1] = y; if $p > 0$ then PATH[1] = x;

(b) PATH[1],$\cdots$, PATH[p], STACK[s], STACK[s-2],$\cdots$, STACK[1] is a subsequence of P;

(c) for each odd k with $3 \le k \le s$, the segment of P that joins i = STACK[k] to j = STACK[k-2] is a shortest STACK[k-1]-path from i to j.

We turn now to EVFA. For $\underset{\sim}{W} \in \underset{=}{F}$ and $0 \le k \le n$, let $\underset{\sim}{W}_k$ denote the set of all walks $(x_0, \cdots, x_t) \in \underset{\sim}{W}$ such that $x_s \le k$ when $0 < s < t$. Thus the end nodes of walks in $\underset{\sim}{W}_k$ are unrestricted but all intermediate nodes are in $\{1, \cdots, k\}$. Let $\underset{=}{F}_k = \{\underset{\sim}{W}_k : \underset{\sim}{W} \in \underset{=}{F}\}$. Since $\underset{\sim}{W}_n = \underset{\sim}{W}$ for all $\underset{\sim}{W} \in \underset{=}{F}$, it suffices to prove the following for $0 \le k \le n$:

(10_k) <u>After the k^{th} passage through the main loop of</u> EVFA, <u>conditions</u> (8) <u>and</u> (9) <u>are satisfied for all</u> $x, y \in N$ <u>and</u> $\underset{\sim}{Z} \in \underset{=}{F}_k$.

The proof is by induction on k. Here initialization is regarded as the 0^{th} passage through the main loop and assertion (10_0) is obvious because $\underset{\sim}{Z}_0$ is merely the set of all edges (paths (x_0, x_1)) in $\underset{\sim}{Z}$.

From the argument below it follows that (10_k) holds for all k regardless of the order in which i, j and $(\underset{\sim}{U}, \underset{\sim}{V}, \underset{\sim}{W})$ appear in the main loop. In particular, the main loop could be written as

<u>for</u> $k \leftarrow 1$ <u>until</u> n <u>do</u>

  <u>for each</u> $(\underset{\sim}{U}, \underset{\sim}{V}, \underset{\sim}{W}) \in \underset{=}{T}$ <u>do</u>

    <u>for each</u> $(i, j) \in \{1, \cdots, n\} \times \{1, \cdots, n\}$ <u>do</u> . . . .

That is convenient for programming in some languages with special array-handling capabilities, such as APL.

Now suppose, with $0 \le k \le n$, that (10_{k-1}) holds, and consider the k^{th} passage through the main loop. We note first that if i or j is k then there do not exist $\underset{\sim}{U}, \underset{\sim}{V} \in \underset{=}{F}$ such that $\underset{\sim}{U}\underset{\sim}{V} \subset \underset{\sim}{W}$ and

(d) $M_{\underset{\sim}{W}}[i, j] = -1$ but $M_{\underset{\sim}{U}}[i, k] \neq -1 \neq M_{\underset{\sim}{V}}[k, j]$ or

(e) $L(S_{\underset{\sim}{U}}[i, k], S_{\underset{\sim}{V}}[k, j]) < S_{\underset{\sim}{W}}[i, j]$.

Suppose, for example, that $j = k$. If (d) holds there is no $\underset{\sim}{W}_{k-1}$-path from i to k but there is a $\underset{\sim}{U}_{k-1}$-path U from i to k and there is a $\underset{\sim}{V}_{k-1}$-path V from k to k. But then UV is a $\underset{\sim}{W}_k$-walk from i to k and by condition (4) there is an associated $\underset{\sim}{W}_k$-path P from i to k. Plainly P is, in fact, a $\underset{\sim}{W}_{k-1}$-path, and that is a contradiction. A similar contradiction is derived from (e), using conditions (1) - (4) and the inductive hypothesis. It follows that the k^{th} rows and k^{th} columns of $S_{\underset{\sim}{W}}$, $M_{\underset{\sim}{W}}$, $U_{\underset{\sim}{W}}$ and $V_{\underset{\sim}{W}}$ are unchanged by the k^{th} passage through the main loop. Hence (10_k) holds for all $\underset{\sim}{Z} \in \underset{=}{F}_k$ and $x, y \in N$ with $x = k$ or $y = k$. The case in which $x \neq k \neq y$ remains.

Supposing, still, that $0 < k \le n$ and (10_{k-1}) holds, consider $\underset{\sim}{W} \in \underset{=}{F}$ and $i, j \in N$ with $i \neq k \neq j$. We discuss only the case in which (e) holds at some time during the k^{th} passage, for the other cases (described in terms of (d) and (e)) are similar. Let

$$\mu = \min\{L(S_{\underset{\sim}{U}}[i, k], S_{\underset{\sim}{V}}[k, j]): (\underset{\sim}{U}, \underset{\sim}{V}, \underset{\sim}{W}) \in \underset{=}{T}\},$$

the minimum during the k^{th} passage, and let $(\underset{\sim}{U}', \underset{\sim}{V}')$ be the first pair $(\underset{\sim}{U}, \underset{\sim}{V})$ for which the minimum is attained. Then at the end of the k^{th} passage,

$$S_{\underset{\sim}{W}}[i,j] = \mu;\ M_{\underset{\sim}{W}}[i, j] = k,\ U_{\underset{\sim}{W}}[i, j] = \underset{\sim}{U}',\ V_{\underset{\sim}{W}}[i, j] = \underset{\sim}{V}'\ .$$

Let U be a shortest $\underset{\sim}{U}'_{k-1}$-path from i to k and let V be a shortest $\underset{\sim}{V}'_{k-1}$-path from k to j, whence $L_\lambda(UV) = \mu$. Then $UV \in \underset{\sim}{U}'\underset{\sim}{V}' \subset \underset{\sim}{W}$ and hence UV is

a $\underset{\sim}{W}_k$-walk from i to j. For (10_k) it suffices to show UV is a shortest $\underset{\sim}{W}_k$-path from i to j. Consider an arbitrary shortest $\underset{\sim}{W}_k$-walk W from i to j and an arbitrary associated path $P = (x_o, \cdots, x_t)$. Then $P \in \underset{\sim}{W}_k$ and $L_\lambda(P) \le L_\lambda(W)$, whence of course

$$L_\lambda(P) = L_\lambda(W) \le L_\lambda(UV).$$

The node k appears in P for otherwise it is true at the end of the $(k-1)^{th}$ passage that $S_{\underset{\sim}{W}}[i, j] = L_\lambda(P)$ and then (e) never holds during the k^{th} passage, contrary to hypothesis. With $x_s = k$ there exist $\underset{\sim}{U}$ and $\underset{\sim}{V}$ such that $(x_o, \cdots, x_s) \in \underset{\sim}{U}$, $(x_s, \cdots, x_t) \in \underset{\sim}{V}$ and $(\underset{\sim}{U}, \underset{\sim}{V}, \underset{\sim}{W}) \in \underset{=}{T}$. But then

$$L_\lambda(UV) \le L(S_{\underset{\sim}{U}}[i, k], S_{\underset{\sim}{V}}[k, j] \le L(L_\lambda(x_o, \cdots, x_s), L_\lambda(x_s, \cdots, x_t)) = L(P),$$

whence $L_\lambda(UV) = L_\lambda(W)$ and UV is a shortest $\underset{\sim}{W}_k$-walk from i to j. If UV is not a path it has an associated path that misses k, and that was shown to be impossible.

The Applications.

Though conditions (1)-(4) suffice for the validity of EVFA, some additional conditions aid in verifying condition (4) for specific applications. The function L is said to be nice if in addition to (1) and (2) it satisfies the following two conditions:

(11) each point of rng L is fixed under L; that is, $L(\alpha_1,\cdots,\alpha_t) = L(L(\alpha_1,\cdots,\alpha_t))$;

(12) if $\beta \geq 0$ then $L(\alpha,\beta) \geq L(\alpha) \leq L(\beta,\alpha)$ for all $\alpha \in$ rng L.

Note that each of L_p, L_∞ and L^p is nice.

If W is a walk $(x_0,\cdots,x_t)$ and the proper segment $(x_r,\cdots,x_s)$ of W is a circuit C then C is called an intermediate circuit of W and W_{rs} denotes the walk that remains when all of C but x_r or x_s is removed from W. More precisely, when $0 < r$ W_{rs} is the walk $(x_0,\cdots,x_r,x_{s+1},\cdots,x_t)$, $(x_0,\cdots,x_r,x_{s+1})$ or $(x_0,\cdots,x_r)$ according as $s+1<t$, $s+1 = t$ or $s = t$, and when $s < t$ W_{rs} is the walk $(x_0,\cdots,x_{r-1},x_s,\cdots,x_t)$, $(x_{r-1},x_s,\cdots,x_t)$ or $(x_s,\cdots,x_t)$ according as $r-1>0$, $r-1 = 0$ or $r = 0$.

Note that a walk is a path if and only if it has no intermediate circuit. Hence condition (4) can be deduced from repeated application of the following condition:

(13) if $W = (x_0,\cdots,x_t) \in \underset{\sim}{W} \in \underset{=}{F}$ and $(x_r,\cdots,x_s)$ is an intermediate circuit of W then $W_{rs} \in \underset{\sim}{W}$ and $L_\lambda(W_{rs}) \leq L_\lambda(W)$.

Note that the inequality of (13) always holds when $L = L_\infty$. In other cases it can often be deduced from the following result.

(14) If L is nice, $C = (x_r,\cdots,x_s)$ is an intermediate circuit of a walk $W = (x_0,\cdots,x_t)$ and $L_\lambda(C) \geq 0$ then $L_\lambda(W_{rs}) \leq L_\lambda(W)$.

To prove (14), note that $W = CW_{rs}$ if $r = 0$, $W = W_{rs}C$ if $s = t$, and if $0<r<s<t$ there are walks U and V such that $W = UCV$ and $W_{rs} = UV$. We consider

only the third case for the others are similar to it. With $L_\lambda(C) \geq 0$, it follows from (11), (12) and (1) that

$$L_\lambda(U) = L(L_\lambda(U)) \leq L(L_\lambda(U), L_\lambda(C)) = L_\lambda(UC),$$

then from (1) and (2) that

$$L_\lambda(W_{rs}) = L_\lambda(UV) = L(L_\lambda(U), L_\lambda(V)) \leq L(L_\lambda(UC), L_\lambda(V)) = L_\lambda(UCV) = L_\lambda(W).$$

Below are some illustrative problems on shortest restricted paths. In each case, "find shortest paths" means that for each $x, y \in N$, either a shortest path from x to y (among those satisfying the indicated restrictions) must be found or it must be concluded that no path from x to y satisfies the restrictions.

(A) A set of edges is given. For each $k \leq \ell$, find shortest paths that use at most k of the special edges.

(B) A set of nodes is given. For each $k \leq \ell$, find shortest paths that use at most k of the special nodes.

(C) A sequence of s sets is given, each consisting of nodes or edges or a mixture. For each choice of $(k_1, \cdots, k_s)$ with $k_r \leq \ell_r$ for all r, find shortest paths that use (for all r) at most k_r of the elements of the r^{th} set.

(D) In addition to the R^*-valued edge-lengths $\lambda(x, y)$, integer edge-lengths $\pi(x,y) \geq 0$ are given. Each walk has its usual length L_λ and also a length I_π where I is L_1. An integer $\ell \geq 0$ is given. For each $k \leq \ell$, find L_λ-shortest paths P subject to the restriction that $I_\lambda(P) \leq k$.

(E) The nodes of G are partitioned into two disjoint sets A and B, and an integer $\ell \geq 0$ is given. For each $k \leq \ell$, find shortest paths that oscillate at most k times between A and B.

(F) A subgraph H of G and an integer $\ell \geq 0$ are given. For each $k \leq \ell$, find shortest paths P for which $P \cap H$ has at most k components.

(G) A set M of edges of G is given. Find shortest M-alternating paths.

As can be seen by reference to conditions (1)-(4) and (13)-(14), *the discussions of* (A)-(G) *below are valid* (*that is*, EVFA *can be applied for the stated purpose*) *if* L *is* L_∞ *and also if* L *is nice and* $L_\lambda(C) \geq 0$ *for each circuit* C *intermediate to a walk belonging to a member of* $\underline{\underline{F}}$, *where* $\underline{\underline{F}}$ *is the family of sets of walks used for the particular problem*. (*Problem* (G) *requires an additional condition*, *stated later*.)

(A) This problem, which was mentioned earlier, is straightforward. For $0 \leq k \leq \ell$, let $\underset{\sim}{W}(k)$ denote the set of all walks $(x_0, \cdots, x_t)$ such that the edge (x_{i-1}, x_i) is special for at most k values of i. Let $\underline{\underline{F}} = \{\underset{\sim}{W}(k) \colon 0 \leq k \leq \ell\}$. Let $\underline{\underline{T}} = \bigcup_{0=k}^{\ell} \underline{\underline{T}}_k$ where

$$\underline{\underline{T}}_k = \{(\underset{\sim}{W}(i), \underset{\sim}{W}(j), \underset{\sim}{W}(k)) \colon i+j = k\}.$$

Then $|\underline{\underline{T}}_k| = k+1$ and $|\underline{\underline{T}}| = (\ell+1)(\ell+2)/2$. The overall time-complexity of EFWA for this problem is $O(\ell^2 n^3)$.

(B) This problem is similar to (A), but it is included to illustrate the way in which the end behavior of walks must sometimes be considered in constructing $\underline{\underline{F}}$ and $\underline{\underline{T}}$ for the application of EFWA. For $0 \leq k \leq \ell$ let $\underset{\sim}{W}_{--}(k)$ $\langle$resp. $\underset{\sim}{W}_{-+}(k)$, $\underset{\sim}{W}_{+-}(k)$, $\underset{\sim}{W}_{++}(k)\rangle$ denote the set of all walks $(x_0, \cdots, x_t)$ such that the node x_i is special for at most k values of i with $0 < i < t$ and, in addition, neither x_0 x_t is special $\langle$resp. x_t is special but x_0 is not, x_0 is special but x_t is not, both x_0 and x_t are special$\rangle$. Let $\underline{\underline{F}}$ consist of the sets $\underset{\sim}{W}_{--}(k)$ for $k \leq \ell$, the sets $\underset{\sim}{W}_{-+}(k)$ and $\underset{\sim}{W}_{+-}(k)$ for $k \leq \ell-1$, and the sets $\underset{\sim}{W}_{++}(k)$ for

$k \le \ell - 2$. Let $\underline{\underline{T}} = \bigcup_{o=k}^{\ell} T_k$, where $\underline{\underline{T}}_k$ consists of the triples

$$(W_{--}(i),\ W_{--}(j),\ W_{--}(k)) \qquad \text{for } i+j = k,$$

$$(W_{--}(i),\ W_{-+}(j),\ W_{-+}(k-1)), \qquad (W_{+-}(i),\ W_{--}(j),\ W_{+-}(k-1))$$

$$\text{and} \qquad (W_{-+}(i),\ W_{+-}(j),\ W_{--}(k)) \qquad \text{for } i+j = k-1,$$

$$(W_{-+}(i),\ W_{++}(j),\ W_{-+}(k-1)), \qquad (W_{++}(i),\ W_{+-}(j),\ W_{+-}(k-1))$$

$$\text{and} \qquad (W_{+-}(i),\ W_{-+}(j),\ W_{++}(k-2)) \qquad \text{for } i+j = k-2,$$

$$\text{and} \qquad (W_{++}(i),\ W_{++}(j),\ W_{++}(k-2)) \qquad \text{for } i+j = k-3.$$

Again, $|\underline{\underline{T}}|$ is $O(\ell^2)$ and the complexity of EVFA is $O(\ell^2 n^3)$.

(C) This is included to illustrate the application of EVFA when the desired paths are subject to several restrictions. In order to avoid a notational morass, only the case of sets of edges is discussed. For $k_1 \le \ell_1, \cdots, k_s \le \ell_s$, let $W(k_1, \cdots, k_s)$ denote the set of all walks $(x_0, \cdots, x_t)$ such that, for $1 \le r \le s$, (x_{i-1}, x_i) belongs to the r^{th} set of edges for at most k_r values of i. Let $\underline{\underline{T}}$ consist of all triples

$$(W(i_1, \cdots, i_s),\ W(j_1, \cdots, j_s),\ W(k_1, \cdots, k_s))$$

such that for $1 \le r \le s$, $i_r + j_r = k_r \le \ell_r$. Then

$$|\underline{\underline{T}}| = \sum_{k_1=0}^{\ell_1} \sum_{k_2=0}^{\ell_2} \cdots \sum_{k_s=0}^{\ell_s} ((k_1+1)(k_2+1) \cdots (k_s+1)) = 2^{-s} \prod_{r=1}^{s} (\ell_r+1)(\ell_r+2)$$

and the complexity of EVFA is

$$O(2^{-s}(\ell_1 \ell_2 \cdots \ell_s)^2 n^3).$$

(D) This may be regarded as the integer-weighted version of a problem of which (A) is the cardinality-weighted version. Similar extensions are available for the

other problems considered here. For $0 \le k \le \ell$, let $\underset{\sim}{W}(k)$ be the set of all walks W for which $I_\pi(W) \le k$. Define $\underline{\underline{F}}$ and $\underline{\underline{T}}$ in the obvious ways. The complexity of EVFA is $O(\ell^2 n^3)$. For a closely related treatment of this problem and of (A), see the discussion of the Bellman-Ford method in [L, pp. 74-75, 92-93]. EVFA is similar to the Bellman-Ford method but is more general. Roughly speaking, it amounts to replacing the additive semigroup $\{0,1,2,\ldots\}$ of Bellman-Ford by an arbitrary semigroup.

(E) This is a special case of a more general problem, which may be formulated as follows: A function ϕ is defined on a set of nodes and edges of G, with $\operatorname{rng} \phi \subset \{1,\cdots,m\}$, and an integer $\ell \ge 0$ is given. For each $k \le \ell$, find shortest paths along which ϕ has at most k relative extrema.

As the term is used here, a <u>relative extremum</u> of a real sequence $(\alpha_0,\cdots,\alpha_u)$ is an ordered pair (r, s) such that $0<r\le s<u$ and

$$\alpha_{r-1} < \alpha_r = \cdots = \alpha_s > \alpha_{s+1} \quad \text{or} \quad \alpha_{r-1} > \alpha_r = \cdots = \alpha_s < \alpha_{s+1}.$$

For a walk $W = (x_0,\cdots,x_t)$, let

$$\exp W = (x_0, (x_0, x_1), x_1,\cdots,(x_{t-1},x_t), x_t),$$

the expanded version of W in which nodes and edges alternate. Let W_ϕ denote the sequence of ϕ-values corresponding to the elements of $\exp W$ that belong to $\operatorname{dmn} \phi$, and let $\rho_\phi(W)$ denote the number of relative extrema of W_ϕ. The general problem is to find shortest paths P for which $\rho_\phi(P) \le k$. Problem (E) is the special case in which

(*) $m = 2$ and $\operatorname{dmn} \phi = N = A \cup B$, with $\phi = 1$ on A and $\phi = 2$ on B.

As is shown below, this can be handled by EVFA. However, we do not know how to use EVFA efficiently for the general problem, or even for the following special cases:

$m = 3$ and dmn $\phi = N$;

$m = 2$ and dmn ϕ is a proper subset of N;

$m = 2$ and dmn ϕ is the set E of all edges of G.

In each case there is difficulty, even when L is L_1 and all values of the edge-length λ are positive, in constructing a suitable family $\underset{=}{F}$ satisfying conditions (3) and (4).

Now let us return to (E) in the formulation provided by (*), except that the condition dmn $\phi = N$ may be replaced by dmn $\phi \supset N$. For $0 \le k \le \ell$ and $u, v \in \{1,2\}$, let $\underset{\sim}{W}_{uv}(k)$ denote the set of all walks $W = (x_0, \cdots, x_t)$ such that $\phi(x_0) = u$, $\phi(k_t) = v$, and W_ϕ has at most k relative extrema (equivalently, W oscillates at most k times between A and B). Let

$$\underset{=}{F} = \{W_{u,v}(k) \colon u, v \in \{1, 2\}, k \le \ell\}$$

and let $\underset{=}{T}$ consist of all triples.

$(\underset{\sim}{W}_{uu}(i), \underset{\sim}{W}_{uu}(j), \underset{\sim}{W}_{uu}(k))$ for $u \in \{1,2\}$ and $(\{i,j\} = \{0,k\}$ or $(i>0<j$ and $i+j = k-1))$,

$(\underset{\sim}{W}_{uv}(i), \underset{\sim}{W}_{uu}(j), \underset{\sim}{W}_{uu}(k))$ for $\{u,v\} = \{1,2\}$ and $i+j = k-1$,

$(\underset{\sim}{W}_{uv}(i), \underset{\sim}{W}_{vv}(j), \underset{\sim}{W}_{uv}(k))$ for $\{u,v\} = \{1,2\}$ and $((i,j) = (k,0)$ or $(j>0$ and $i+j = k-1))$,

$(\underset{\sim}{W}_{uu}(i), \underset{\sim}{W}_{uv}(j), \underset{\sim}{W}_{uv}(k))$ for $\{u,v\} = \{1,2\}$ and $((i,j) = (0,k)$ or $(i>0$ and $i+j = k-1))$.

Then EVFA can be applied, solving problem (E) in time $O(\ell^2 n^3)$.

(F) Define ϕ on all nodes and edges of G, with $\phi = 1$ on nodes and edges of the graph H and $\phi = 2$ otherwise. With the $\underset{\sim}{W}_{uv}$ as in the preceding paragraph, the paths P for which $P \cap H$ has at most k components are precisely the paths in

$$\underset{\sim}{W}_{11}(2k-3) \cup \underset{\sim}{W}_{12}(2k-2) \cup \underset{\sim}{W}_{21}(2k-2) \cup \underset{\sim}{W}_{22}(2k-3).$$

Hence (F) can also be handled by EVFA in time $O(\ell^2 n^3)$.

(G) This problem is also discussed in a more general setting. With ϕ as in the discussion of (E) and with $m = 2$, let a walk W be called ϕ-alternating if the sequence W_ϕ alternates between 1 and 2. How can shortest ϕ-alternating paths be found? Problem (G) is the special case in which $\text{dmn}\ \phi = E$, $\phi = 1$ on M, and $\phi = 2$ on $E \sim M$.

When $\text{dmn}\ \phi \supset N$, EVFA can be applied by taking $\underset{=}{F} = \{\underset{\sim}{W}_{11}, \underset{\sim}{W}_{12}, \underset{\sim}{W}_{21}, \underset{\sim}{W}_{22}\}$, where $\underset{\sim}{W}_{uv}$ is the set of all ϕ-alternating walks W such that the sequence W_ϕ starts with u and ends with v. Then let $\underset{=}{T}$ consist of all triples $(\underset{\sim}{W}_{uv}, \underset{\sim}{W}_{vu}, \underset{\sim}{W}_{uu})$ and $(\underset{\sim}{W}_{uv}, \underset{\sim}{W}_{vv}, \underset{\sim}{W}_{uv})$ for all $u, v \in \{1,2\}$.

Now consider the case in which $\text{dmn}\ \phi = E$. Then each pair (i, j) forms a ϕ-alternating path (recall the standing hypothesis that G is the complete graph on N) but of course we are interested only in paths of finite length. Define $\underset{=}{F}$ by restricting the $\underset{\sim}{W}_{uv}$'s of the preceding paragraph to include only walks of finite length, and assume

($\dagger$) each alternating circuit of finite length has an even number of edges.

Then problem (F) can be handled by EVFA with $\underset{=}{T}$ consisting of all triples $(\underset{\sim}{W}_{up}, \underset{\sim}{W}_{qv}, \underset{\sim}{W}_{uv})$ for $u, v \in \{0,1\}$ and $\{p,q\} = \{0,1\}$. However, this approach may fail when ($\dagger$) fails for then a path associated with an alternating walk need not be an alternating path and thus condition (4) may fail. For example, consider Fig. 1 and note that (x_2, X_3, X_4, x_5) is an alternating circuit according to our definition, where the solid edges are those in M.

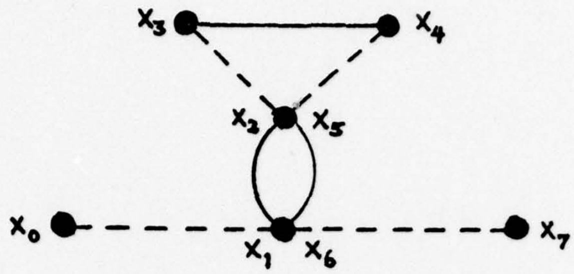

Fig. 1: The walk $(x_0,\cdots, x_7)$ is alternating but the associated path (x_0,x_1,x_7) is not.

Brown [B] suggests a method for finding shortest M-alternating paths in a directed graph $D = (N, E)$ (no longer assumed complete). Another directed graph D^* is constructed, having two nodes x' and x'' for each node x of D, and the edges of D^* are obtained as follows for each edge (x, y) of D with length $\lambda(x,y)<\infty$.

when $(x,y) \in M$, (x',y'') is an edge of D^* with length $\lambda(x, y)$;

when $(x,y) \notin M$, (x'', y') is an edge of D^* with length $\lambda(x, y)$.

It is claimed [B] there is a natural one-to-one correspondence between alternating paths in D and ordinary paths in D^*. Thus the problem of finding shortest alternating paths in D is equivalent to the problem of finding shortest ordinary paths in D^*. The claim is correct when (+) holds but not in general, as can be seen from Fig. 1. For example, if D is the graph of Fig. 1 then the path $(x_0'',x_1',x_2'',x_3',x_4'',x_2',x_1'',x_4')$ in D^* corresponds to the walk $(x_0,x_1,x_2,x_3,x_4,x_2,x_1,x_7)$ in D.

In general, Brown's construction does produce a one-to-one correspondence between the walks in D^* and the alternating walks in D. If D has no negative alternating circuit then D^* has no negative circuit and the Floyd-Warshall algorithm can be applied to find shortest paths (=shortest walks) in D^* and hence alternating walks in D. The latter may or may not be paths. For general graphs, even when $L = L_1$ and may or may not be paths. For general graphs, even when $L = L_1$ and $\lambda > 0$, we

do not know how to apply EFVA directly to find shortest alternating paths. However, the more complicated "blossom" methods of Edmonds [E1][E2][L] will apparently apply to this problem.

We close with a query. For $0 \le \ell \le m < n$ and for $\rho \in \{\le, =, \ge\}$ let $\underline{\underline{P}}_0(\ell,m,n,\rho)$ $\langle$resp. $\underline{\underline{P}}_1(\ell,m,n,\rho)\rangle$ denote the following problem:

A complete graph G is given, with n nodes and positive edge-lengths. In addition, a set of m special nodes $\langle$resp. edges$\rangle$ of G is given. Find shortest paths P in G such that the number of special nodes $\langle$resp. edges$\rangle$ used by P is in the relation ρ to ℓ.

We have seen here that the extended version of Floyd's algorithm solves $\underline{\underline{P}}_0(\ell,m,n,\le)$ and $\underline{\underline{P}}_1(\ell,m,n,\le)$ in time $O(\ell^2 n^3)$. By contrast, the problem $\underline{\underline{P}}_0(n,n,n,=)$ is essentially the traveling salesman problem and hence is NP-complete [K1][K2][AHU]. What else of interest can be said about the computational complexity of $\underline{\underline{P}}_0(\ell,m,n,\rho)$ and $\underline{\underline{P}}(\ell,m,n,\rho)$ for $\rho \in \{=, \ge\}$?

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In a directed network with no negative circuit, Floyd's algorithm finds, for each pair of nodes x and y, a shortest path from x to y. Here the procedure is extended to minimize more general length functions over sets of paths that are restricted in various ways.

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